Exploring CONCEPTS in PROBABILITY using graphics calculators

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World over, hand-held technology in the form of graphics calculators has changed the scenario of mathematics teaching. The past two decades have witnessed extensive research related to the use of hand-held technology for the teaching and learning of mathematics (Demana & Waits, 1998; Jones, 1995). Mathematics educators in many countries are concerned with issues such as: the role of paper-pencil skills in a teaching environment equipped with hand-held technology, replacing paper-pencil algorithms with graphics calculator technology while still retaining some basic by-hand skills, reforming the curriculum to make appropriate use of hand-held technology and finding alternative modes of assessment that incorporate graphics calculator technology (Demana &Waits, 1992a, 1992b). Researchers claim that calculators lead to improved problem solving because they free more time for instruction, provide more tools for problem solving, allow students to focus on concepts, and free the students from the burden of computation (Dunham, 1999).

This article describes a project in which certain key concepts in probability were explored using graphics calculators with year 10 students. The lessons were conducted in the regular classroom where students were provided with a Casio CFX 9850 GB PLUS graphics calculator with which they were familiar from year 9. At the school where I teach in India, using technology for teaching is not a regular practice. Being in-charge of an experimental project funded by the Department of Education (one of whose objectives is to explore the use of technology for teaching mathematics), however, I have had the opportunity to integrate technology with teaching.

Educational setting and background knowledge of students

The participants in the activities were forty-five students in Year 10. In the curriculum (as prescribed by the Central Board of Secondary Education in India) Probability forms a part of the syllabus in Year 10. In the classroom each student was given a Casio CFX 9850 GB PLUS graphics calculator. Before the graphics calculator lessons, the students had acquired the following concepts and skills through regular classroom teaching:

(i) sample space, random experiment, and theoretical probability of an event;

(ii) computation of the probability of events using the basic definition of probability.

Students, however, had varied notions of the concept of randomness. When asked what randomness meant, responses included the following:

- ‘Randomness means something occurring in no definite pattern.’
- ‘Randomness means that when we have two outcomes, they have an equal chance of occurring.’
- ‘Randomness means no fixed occurrence of an “object”.’

- 'Randomness means no fixed chance.'
- 'Randomness refers to the unpredictable manner in which an event can occur. If there are many possibilities each one has the same chance of occurring.'

Aim of project

The primary objective of the project was for students to explore some basic concepts in probability using graphics calculators in the regular traditional classroom environment. In particular the aims were to:

- introduce the concept of randomness;
- emphasise the difference between empirical and theoretical probabilities of an event;
- introduce the concept of simulation and highlight its importance; and
- simulate experiments such as the throwing of coin(s) or dice and the birthday problem using a graphics calculator.

Ideally the graphics calculator lessons should have been conducted before formal probability theory was introduced in the regular classroom. Since using technology for teaching is not a regular practice, however, several issues had to be considered. At the school where I teach, there are 12 sections of year 10 with class sizes ranging between 45 and 50. The 45 students who participated in this project were selected from across these 12 sections. Most of these students were members of a mathematics lab and were familiar with the Casio CFX 9850GB PLUS graphics calculator. It was decided by the mathematics staff that the selected students would participate in the project only after they had undergone the regular classroom lessons with the rest of their classmates so that the regular schedule of teaching was not disturbed.

Graphics calculator lessons

During the technology lessons a worksheet was given to each student, which explained the following concepts in a step-by-step manner:

- basic definition of probability and meaning of randomness;
- random experiment and sample space;
- probability of an event;
- theoretical probability versus empirical probability of an event;
- simulation.

Although much of this had already been taught in the regular classes it was felt that there were 'gaps' in the students' understanding of some of the concepts and a revision was essential. In the worksheet, each concept was followed by exercises, which enabled the students to explore the concept either by simulating an experiment on the calculator or by actually performing the experiment and recording the observations. Some exercises required by-hand calculations.

Concept of randomness

After defining probability as, 'a mathematical model for measuring the uncertainty of an event', the concept of randomness was introduced. An exercise required the students to generate numbers randomly in the **RUN** mode of the graphics calculator by entering the **Ran#** function, which produced a pseudo-random number between 0 and 1 each time **EXE** was pressed, as shown in Figure 1.

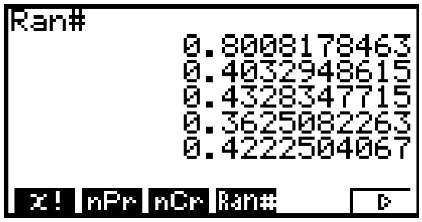

Figure 1

The following exercise required students to generate 10 integers between 1 and 10 randomly and repeat this experiment 10 times, each time counting the number of integers greater than 5. This was done in the **TABLE** mode by entering **Int(10×Ran# + 1)** and setting the Range as 1 to 100, as shown in Figure 2.

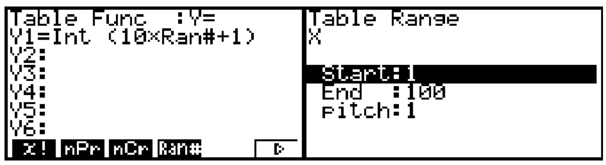

Figure 2

Figure 3 shows the first few lines of one student's screen output of integers. Her results have been tabulated in Table 1.

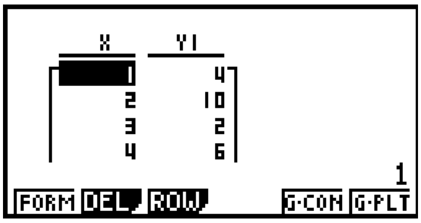

Figure 3

Table 1. Summary of output from random generation of numbers from 1 to 10.

Experiment	Number of integers > 5
1	5
2	5
3	5
4	7
5	5
6	5
7	7
8	5
9	6
10	4
TOTAL	54

For this particular student, out of 100 randomly generated integers between 1 and 10, 54 were greater than 5. Different students obtained different answers. Combining the answers of the entire class revealed that the fraction of integers greater than 5 was very close to 1/2. I emphasised the relevance of this experiment, by explaining that the chances (or probability) of a randomly selected integer between 1 and 10, being greater than 5, was equal to 1/2.

Random experiment and sample space

Having dealt with randomness, the concepts of random experiment, outcomes of a random experiment and sample space were introduced. The tossing of coin(s) and throws of a die were introduced as examples. An exercise required the students to write the sample space for the simultaneous toss of two, three, and four coins and to generalise by concluding that the number of outcomes for n coins is equal to $2n$. A similar exercise required the students to make a generalisation for n die. All students were able to arrive at these generalisations without much effort.

Theoretical probability of an event

In the next part of the worksheet the theoretical probability of the occurrence of an event, E, was defined as, 'Number of outcomes favourable to the event E / Total number of outcomes in the sample space'. This was explained via various examples and an exercise required the students to compute the theoretical probabilities of events such as getting at least one tail in the toss of three coins or getting a sum of 7 in the throw of a pair of dice.

Difference between the theoretical and empirical probability of an event

After covering the theoretical definition, the difference between the theoretical probability and the empirical probability of an event was explained by citing the example of tossing a single coin where the theoretical probability of obtaining 'heads' is 0.5. As I explained however, tossing a coin 100 times may not result in exactly 50 'heads.' Instead one may obtain 47 heads, in which case the empirical probability of getting 'heads' is 0.47. I emphasised that although the empirical probability (also called 'relative frequency') gives the actual fraction of the total trials that result in the occurrence of an event, the theoretical probability gives the 'long-term' fraction of the total trials that result in the occurrence of that event. An exercise (designed to highlight this difference) required each student to perform the experiment of throwing a die 100 times and computing the empirical probability of obtaining a 1 (using the formula: Number of 1s /100). Further students were required to combine their results with those of four others in the class to obtain 500 throws in all and then to compute the empirical probability by successively adding the number of 1s obtained in each successive 100 throws. They were asked to record their observations in a table and plot the empirical probabilities versus the number of throws. One student's observations are shown in Table 2 and Figure 4 respectively. These graphs offered a good visual way of convincing the students that as the number

Table 2. Empirical probability of getting 1 in the throw of a die obtained by progressively adding the number of 1s in five sets of 100 throws each.

Numbers of throws	100	200	300	400	500
Successive number of 1s	23	38	60	70	87
Progressive Empirical Probability	0.23	0.19	0.20	0.18	0.17

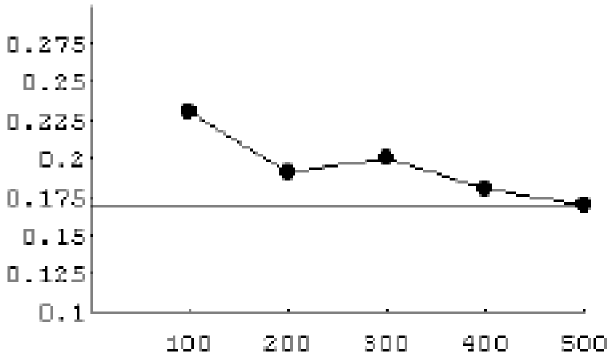

Figure 4

of throws was increased, the empirical probability came closer to the theoretical probability of 1/6 or 0.17.

Simulation on the graphics calculator

In the following lab session, the concept of simulation was introduced. Simulation was described as the process of estimating the probability of an event by running an experiment a large number of times. It was pointed out that since it may be impractical to run the actual experiment, it could be simulated by a random device such as a pseudo-random number generator in a computer or a graphics calculator. Although calculators and computers cannot generate purely random numbers, pseudo-random numbers are sufficient for simulating experiments in the classroom. In a simulation experiment a large number of trials are generated and the relative frequencies of the required event are used to estimate the probability of that event. To estimate the probability of obtaining a 6 in the throw of a dice, for example, one could actually throw a die a large number of times and use the relative frequency of sixes (number of sixes / total number of throws) to estimate the probability or one could simulate the throws by generating the numbers 1 to 6, randomly using the **Ran#** function on the graphics calculator.

Simulation activities

Simulating the throw of a die

An exercise required each student to simulate 100 throws of a die on the graphics calculator by randomly generating 100 integers from 1 to 6 in the **TABLE** mode by typing **Int(6×Ran#+1)**, as shown in Figure 5. Once the integers were generated they were stored in the **LIST** mode and sorted. Having done this the students counted the number of 1s, 2s, 3s, 4s, 5s, and 6s and computed the outcomes' respective empirical probabilities. They appreciated the advantages of simulating the dice throwing experiment on the calculator since it took far less time than actually throwing the dice!

Figure 5

Simulating the toss of a pair of coins

Another exercise required students to simulate 100 tosses of a pair of coins by randomly generating 1s and 2s (1 for 'heads' and 2 for 'tails'). In the **TABLE** mode **Int(2×Ran# + 1)** was entered in **Y1** and **Y2**. The numbers were stored in **List1** and **List2** respectively, as seen in Figure 6.

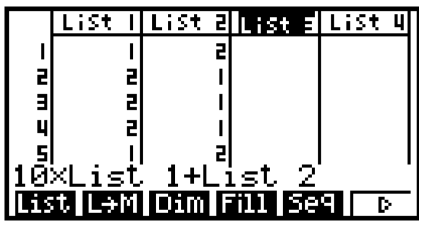

Figure 6

In **List3**, **10** × **List1** + **List2** was entered. This generated a list of 11s (two heads), 12s, 21s (one head and one tail) and 22s (two tails),

as shown in Figure 7. **List3** was then sorted to make the counting easier. Students were asked to record their observations in a table. The results obtained by one particular student have been entered in Table 3.

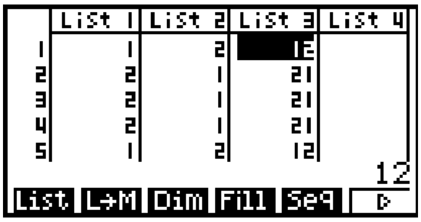

Figure 7

Table 3. Empirical and theoretical outcomes for tossing two coins.

	2 heads	1 head and 1 tail	2 tails
Empirical probability	0.19	0.48	0.31
Theoretical probability	0.25	0.50	0.25

Table 4. Summary of simulation for achieving a 7 when tossing two dice.

(1,6)	(2,5)	(3,4)	(4,3)	(5,2)	(6,1)
2	2	3	1	1	2
Empirical probability of obtaining a sum of 7 = 0.11					

Simulating the throw of a pair of dice

The next exercise set the following tasks:

> Compute the theoretical probability of obtaining a sum of 7 in the simultaneous throw of a pair of dice. Simulate 100 throws on the graphics calculator and compute the empirical probability.

Students concluded that a sum of 7 could be obtained if the pair of die showed the combinations (1,6), (2,5), (3,4), (4,3), (5,2), and (6,1) (out of 36 possible combinations), leading to a theoretical probability of 1/6. 100 throws were simulated by entering **Int(6×Ran# + 1)** in **Y1** and **Y2** in the **TABLE** mode. This generated two lists of random integers between 1 and 6, which were stored in **List1** and **List2** respectively in the **LIST** mode. Students suggested storing **List1 + List2** in **List3**, sorting and counting the number of sevens to compute the empirical probability of obtaining a sum of 7. One student's outcomes from 100 simulated tosses are given in Table 4.

Simulating the birthday problem

In the subsequent lab session students were asked to consider the following question:

> How many people do you need in a group to ensure that the probability of at least two of them having the same birthday is about half? Can you explain your answer?

The estimates provided an interesting range of group sizes. Some were as low as 15 whereas others were as high as 365 or 366! Only two students gave the correct answer 23 but no explanations were provided. In order to verify that the answer is indeed 23, each student was asked to write down ten different birthdays (of friends or relatives) on different slips of paper, which were folded and put in a box. Ten sets of 23 birthdays each were created from the contents of the box. Students working in threes were assigned one set each and had to check for a repeated birthday. Five groups out of 10 had at least one repeat (2 out of these had two repeats) the first time. This experiment was repeated twice. The second time 5 sets showed at least one repeated birthday and the third time 4 sets showed the same. This somewhat convinced the students that the probability of finding at least one repeated birthday in a group of 23 people was about half!

The next step was to simulate the problem on the graphics calculator. Each student had to generate twenty-three birthdays randomly on the calculator using the following steps.

Step 1: Generate a list of 23 integers randomly between 1 and 12 (including 1 and 12) to denote the month by entering **Int(12×Ran# + 1)** in **Y1** in the **TABLE** mode and setting the Range as 1 to 23.

Step 2: Generate a list of 23 integers randomly between 1 and 31 (including 1 and 31) to denote the day of the month by entering **Int(31×Ran# + 1)** in **Y2** in the **TABLE** mode and setting the **Range** 1 to 23, as shown in Figure 8.

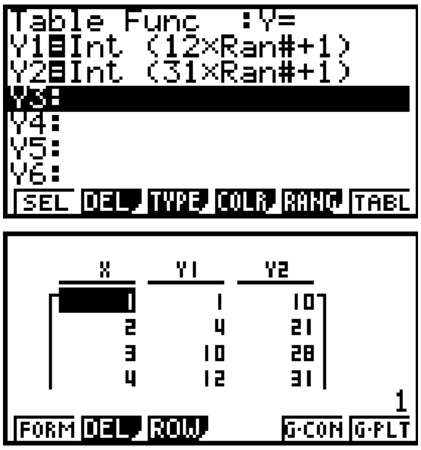

Figure 8

Step 3: Store the column **Y1** into **List1** and **Y2** into **List2**. In the **LIST** mode, take the highlight to **List3**, and type **100** × **List1 + List2**. This converts all the dates to three or four digit numbers whose first one or two digits indicate the month and the last two digits indicate the day of that month. For example, 225 indicates 25 of February and 1019 indicates 19 October. Thus **List3** is a list of 23 randomly generated birthdays (Figure 9). In case the list contains an impossible date such as 31 April, the entire list may be rejected and a new one may be generated. A more efficient method, however, is suggested at the end of this section.

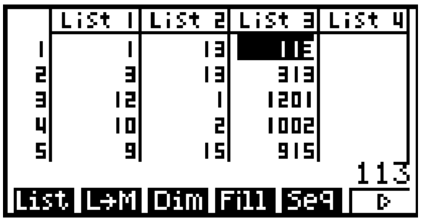

Figure 9

Step 4: Sort **List3** to check for matches. This ensures that the repeated dates appear one after the other, as seen in Figure 10.

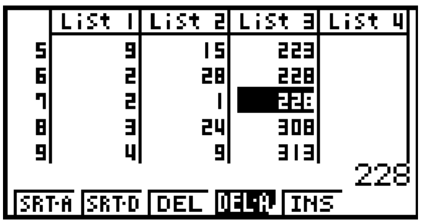

Figure 10

Each student was asked to run the simulation ten times and check for a match each time. In case of a match the repeated birthday was to be recorded. One student's data have been entered in Table 5. Most students obtained a match 5 out of 10 times! Certainly after this experiment the birthday problem became more believable to them!

A simpler and perhaps more efficient way of simulating the birthday problem on the graphics calculator is to generate a list of 23 random numbers between 1 and 365 (representing the 23 birthdays) and sorting the list to check for a repeat. In this method each day of the year is identified by a number between 1 and 365 and there is no risk of obtaining an impossible date.

Table 5. Outcomes for 10 simulations of 23 random birthdays.

Simulation	Repeated birthday/ No match
1	No match
2	228 (28 February)
3	No match
4	604 (4 June)
5	No match
6	No match
7	No match
8	331 (31 March)
9	1004 (4 October)
10	724 (24 July), 1208 (8 Dec.)

Student response

The graphics calculator lessons took four thirty-five minute classes. At the end of these lessons students were asked to fill in a questionnaire where they responded to the statements given in Table 6 by indicating one of the following: Strongly agree (SA), Agree (A), Not Sure (NS), Disagree (D) and Strongly Disagree (SD).

Some general comments on the lessons included the following.

- 'These lessons helped me to explore and understand the basic principles of probability.'
- 'These lessons have been especially helpful in highlighting the difference between the empirical and theoretical probability of an event.'
- 'Teaching this topic through the worksheet and via graphics calculators is a much better way of teaching than the regular classroom methods. It was real fun doing the experiments on the calculator.'

Table 6. Student response to the questionnaire for feedback of the graphics calculator lessons.

Item No.	Item	SA	A	NS	D	SD
1.	Before going through these lessons you had a fairly good idea of the basic concepts of probability.	22	8	9	4	2
2.	Prior to these lessons you had a good idea of the concept of randomness.	—	2	6	13	24
3.	The graphic calculator helped you to explore concepts and verify results on your own.	25	15	5	—	—
4.	Simulating the experiments on the graphic calculator enhanced your understanding in the topic.	18	12	10	3	2
5.	The graphic calculator should be integrated into other topics of the syllabus in a similar way.	33	8	4	—	—
6.	These lessons made the learning of probability more enjoyable than regular traditional classroom teaching.	27	12	6	—	—

- 'The concept of simulation was entirely new to me. I realise its importance since it would be really tiresome to throw a coin or dice 1000 or more times and then compute the empirical probability.'

Discussion

Integrating hand-held technology in a traditional teaching environment served the following purposes:

(i) Students were able to simulate experiments and arrive at results on their own. This gave them a sense of discovery.

(ii) The use of the graphics calculator generated greater enthusiasm and interest among students than traditional classroom teaching.

(iii) The use of calculators made the lessons more interactive. Students gave their own suggestions and asked more questions than they would have in a traditional class.

(iv) The calculator helped to estimate the probability of an event by enabling the students to generate a large number of trials by simulation. It also helped to highlight the difference between the theoretical and empirical probability of an event.

The students' feedback revealed that a majority of them wanted the calculator to be integrated with other topics of the curriculum. This was particularly encouraging since it paved the way for further technology use in other topics of the curriculum. It may be added here that all experiments conducted using the Casio CFX 9850 GB PLUS graphics calculator in this project can be easily carried out on other brands of graphics calculator as well.

References

Demana, F. & Waits, B. K. (1992a). A computer for all students. *The Mathematics Teacher, 85*, 94–95.

Demana, F. & Waits, B. K. (1992b). The power of visualisation in calculus. In E. D. Laughbaum (Ed.), *HandHeld Technology in Mathematics and Science Education: A Collection of Papers* (pp. 49–67). Columbus, OH: Teachers Teaching with Technology College Short Course Program at The Ohio State University.

Demana, F. & Waits, B. K. (1998). The role of graphing calculators in mathematics reform. In E. D. Laughbaum (Ed.), *HandHeld Technology in Mathematics and Science Education: A Collection of Papers* (pp. 72–75). Columbus, OH: Teachers Teaching with Technology College Short Course Program at The Ohio State University.

Dunham, P. H. (2000). Hand-held calculators in mathematics education: A research perspective. In E. D. Laughbaum (Ed.), *HandHeld Technology in Mathematics and Science Education: A Collection of Papers* (pp. 39–47). Columbus, OH: Teachers Teaching with Technology College Short Course Program at The Ohio State University.

Jones, P. L. (1995). Realising the educational potential of the graphics calculator. In L. Lum (Ed.), *Proceedings of the Sixth Annual International Conference on Technology in Collegiate Mathematics* (pp. 212–217). Reading, MA: Addison-Wesley.

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